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Journal of Applied Mathematics and Mechanics 71 (2007) 485–493
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Adhesive resistance in the rolling of elastic bodies☆

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Received 22 March 2006

Abstract

A model of the rolling of a rough cylinder on an elastic half-space when there is adhesive attraction between the surfaces, due to molecular interaction, is constructed. The contact characteristics and moment of the adhesive resistance to rolling are calculated.

It has been established theoretically and experimentally that the main sources of resistance to rolling is the relative slippage of the surfaces in the contact area, which occurs due to differences in the curvatures and mechanical properties of the contacting bodies,[1,2] the imperfect elasticity of practical materials,[3] and forces of molecular adhesion.[4]

Molecular interaction of the rolling surfaces is a source for the occurrence of the adhesive component of the friction force. A theoretical-experimental estimate of this component of the rolling friction force was carried out in Ref. 4, based on the assumption that each approach and separation of the molecules is accompanied by a loss of energy. The resistance to rolling was also explained in Ref. 5 by the attraction of the separating parts of the surfaces, due to the occurrence in them of unlike electric charges.

In this paper, an estimate of the adhesive component of the friction force, is based on a mechanism for the dissipation of energy when irregularities of contacting rough bodies approach and separate when one body rolls over the surface of the other. When solving the problem, relations obtained earlier in Ref. 6 for the contact characteristics and values of the energy dissipation in an approach-separation cycle of two axisymmetrical elastic asperities are used. The dependence of the adhesion attraction of these surfaces, due to molecular interaction, on the distance between them, is described by a Lennard - Jones potential and is approximated by a step function (the Maugis - Dugdale model). The stress distributions in the region of the contact interaction, the values of the nominal and actual contact areas, and also the moment of the resistance to rolling are investigated for different values of the parameters of the surface roughness and different mechanical and physical characteristics of the interacting bodies.

1. Formulation of the problem

Suppose a rigid cylinder of radius R rolls with an angular velocity ω on an elastic half-space. The problem is considered in a moving system of coordinates, the z axis of which is perpendicular to the cylinder axis and is directed into the depth of the half-space, while the x axis lies in the undeformed surface of the elastic half-space and is directed along the motion of the cylinder. The surface of the cylinder is covered with a periodic system of similar asperities,

☆ *Prikl. Mat. Mekh*. Vol. 71, No. 4, pp. 534–543, 2007.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.09.001

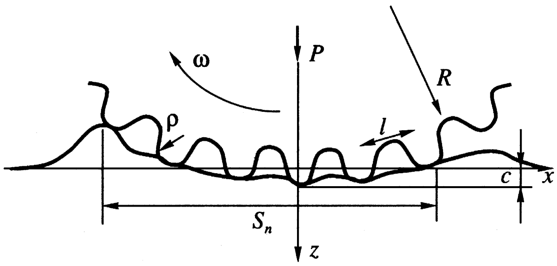

Fig. 1.

situated at the nodal points of a square lattice with pitch l. The tips of the asperities have a spherical shape of radius ρ where $\rho \ll R$. An external normal force P is applied to the cylinder axis (Fig. 1).

The value of the gap between the surfaces of the rough cylinder and the elastic half-space is given by the expression

$$h(x, y) = u(x, y) + f(x, y) - c$$

where $u(x, y)$ is the elastic displacement of the surface of the half-space in the direction of the z axis, $f(x, y)$ is a function describing the shape of the surface of the rough cylinder, and c is the maximum depth of penetration of the cylinder into the elastic half-space.

Contact between the cylinder and the half-space occurs in regions A_i in which the following contact conditions are satisfied

$$h(x, y) = 0 \quad (x, y) \in A_i \tag{1.1}$$

The friction on the contact surface is assumed to be equal to zero.

The surfaces of the cylinder and the elastic half-space experience adhesive attraction in regions B_i, either doubly connected and surrounding the regions of contact of the asperities with the half-space A_i, or simply connected for asperities that are not in direct contact with the half-space. The dependence of the force of adhesive attraction per unit area on the distance between the surfaces is described by a step function (the Maugis-Dugdale model in Ref. 7), so that the pressure $p(x, y)$ on the surface of the elastic half-space in the regions B_i is given by the relation

$$p(x, y) = \begin{cases} -p_0, & h(x, y) \le h_0, \\ 0, & h(x, y) > h_0, \end{cases} \quad (x, y) \in B_i \tag{1.2}$$

In this case the value of the surface energy, which is equal to the work per unit area, performed when the smooth surfaces move away from one another to an infinite distance, is given by the expression

$$\gamma = \int_0^\infty p(h)dh = p_0 h_0 \tag{1.3}$$

2. Method of solution

Reduction to a problem for an individual asperity. In solving the problem it is assumed that the stress-strain state of the elastic half-space in the neighbourhood of each asperity is unaffected by the other asperities. This assumption holds if the distance between the asperities is fairly large.[8]

Consider the position of a rough cylinder for which the lowest asperity is situated symmetrically about the z axis (Fig. 1). Suppose the depth of penetration of this asperity into the elastic half-space, which coincides with the maximum depth of penetration of the whole cylinder, is equal to c, while the depth of penetration of a certain i-th asperity is c_i (Fig. 2). From the triangles ABC and AOC (O is the centre of the cylinder, A is the vertex of the i-th asperity and C is the vertex of the central asperity) we have for the length of the section AC

$$AC = (c - c_i)/\sin(\psi_i/2) = 2R\sin(\psi_i/2)$$

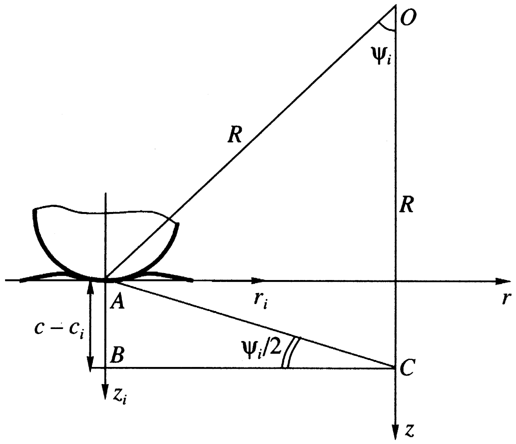

Fig. 2.

Assuming that $l \ll R$, we obtain $\psi_i = il/R$. Then, the penetration of the i-th asperity is given by the relation

$$c_i = c - 2R\sin^2(il/(2R)) \tag{2.1}$$

where i takes values from 1 to N^r on the right of the central asperity, and from 1 to N^l on the left of it. In all, in the interaction with the half-space there are $N^r + N^l + 1$ asperities.

Hence, knowing c_i we can consider the interaction of the i-th asperity with the half-space. We will introduce a local cylindrical system of coordinates (r_i, φ_i, z_i) with origin at the centre of the contact area. Since the asperity has a spherical shape, the distributions of the pressures and elastic displacements in the half-space are symmetrical about the z_i axis in the neighbourhood of an asperity (Fig. 2). Inside the contact area A_i, which is a circle of radius a_i, the contact condition for a displacement u along the z_i axis of the boundary of the elastic half-space is satisfied. By relation (1.1) this condition has the form

$$u(r_i) = -r_i^2/(2\rho) + c_i, \quad r_i \le a_i \tag{2.2}$$

It follows from condition (1.2) and the symmetry of the problem that, in the region of adhesive interaction B_i, which is a ring $a_i < r_i < b_i$, the pressure p on the boundary of the elastic half-space is determined by the adhesive attraction of the surfaces; outside the region of interaction the pressure is equal to zero:

$$p(r_i) = \begin{cases} -p_0, & a_i < r_i < b_i \\ 0, & r_i > b_i \end{cases} \tag{2.3}$$

The tangential stresses on the boundary of the elastic half-space are equal to zero. The normal pressures $p(r_i)$ and displacements $u(r_i)$ are connected by the well-known relation for the axisymmetric contact probem[9]

$$u(r) = \frac{4}{\pi E^*}\int_0^\infty p(r')\mathbf{K}\left(\frac{2\sqrt{rr'}}{r+r'}\right)\frac{r'dr'}{r+r'}, \quad \frac{1}{E^*} = \frac{1-\nu^2}{E} \tag{2.4}$$

where E and ν are Young's modulus and Poisson's ratio of the elastic half-space and $\mathbf{K}(x)$ is the complete elliptic integral of the first kind.

Eq. (2.4) with boundary conditions (2.2) and (2.3) enables us to determine the normal contact pressures $p(x_i)$ and displacements $u(r_i)$ in the region of the contact area of the i-th protuberance with the half-space. To determine the unknown quantities a_i and b_i it is necessary to use the condition of continuity of the pressures at the boundary of the contact area

$$p(a_i) = -p_0 \tag{2.5}$$

and the condition for the surface energy to be constant which, taking relations (1.2) and (1.3) into account, has the form

$$h(b_i)p_0 = \gamma \tag{2.6}$$

where $h(b_i)$ is the value of the gap when $r_i = b_i$

$$h(b_i) = u(b_i) + b_i^2/(2\rho) - c_i$$

Relations (2.2)–(2.6) enable us to determine the contact pressures and elastic displacements of the half-space in the neighbourhood of the i-th asperity, the indentation of which is given by expression (2.1). After this, the normal force acting from the side of the elastic half-space on each asperity, is determined by the expression

$$q_i = 2\pi\int_0^{b_i} rp(r)dr \tag{2.7}$$

Solution of the problem for an isolated asperity. Using the solution of the problem of the interaction of an axisymmetric punch with an elastic half-space when there is adhesive attraction, specified in the form of a step function,[7] for adhesive contact of the asperity with an elastic half-space we have the following equation relating the radii of the contact areas and the adhesive interaction a_i and b_i

$$\frac{4b_ip_0^2}{\pi E^*}(1-\alpha_i+\varphi_i\sqrt{1-\alpha_i^2}) - \frac{b_i^2p_0}{\rho\pi}[(2\alpha_i^2-1)\varphi_i+\alpha_i\sqrt{1-\alpha_i^2}]+\gamma = 0 \tag{2.8}$$

and also expressions for the depth of penetration of the asperity and the forces acting on it

$$c_i = -2b_i\left(\frac{p_0}{E^*}\sqrt{1-\alpha_i^2}+\alpha_i^2\right), \quad q_i = \frac{4}{3\rho}E^*b_i^3\alpha_i^3 - p_0b_i^2\left(2\alpha_i\sqrt{1-\alpha_i^2}-\frac{\varphi_i}{2}\right) \tag{2.9}$$

Here

$$\alpha_i = a_i/b_i, \quad \varphi_i = -\arccos\alpha_i$$

Solving Eq. (2.8) for b_i and substituting the result into relations (2.9) we obtain the relation between the normal force q_i, acting on the asperity, and the depth of indentation of the asperity c_i in the form of a function of the parameter α_i.

Interaction of the asperity with the half-space may also occur when there is no direct contact between the surfaces. In this case a negative adhesive pressure-p_0 acts on the half-space in a circular area of radius b_i. When solving this problem, contact condition (2.2) is not used, and it is assumed that $a_i = 0$ in the remaining conditions. As a result, for the case when there is no contact, we obtain the relations

$$c_i = \frac{b_i^2}{2\rho} - \frac{4p_0b_i}{\pi E^*} - \frac{\gamma}{p_0}, \quad q_i = -\pi b_i^2 p_0 \tag{2.10}$$

The dependence of the normal force q_i, acting on the asperity from the side of the elastic half-space, on the depth of penetration c_i has the form shown in Fig. 3. The heavy curve corresponds to contact of the surfaces and relations (2.8) and (2.9), while the thin curve is for the case when there is no contact and relations (2.10). It can be seen that the dependence of the force on the depth of penetration is described by a non-unique function. According to this relation, as the depth of penetration c_i increases from -∞ (as the asperity approaches the elastic half-space) the surfaces intermittently interact when $c_i = c^r$; when the depth of penetration increases further the interaction is described by curve *1*. We will denote the dependence of the force on the depth of penetration, corresponding to curve *1*, by $q_i^r(c_i)$. When the depth of penetration decreases from the maximum value c (when the asperity withdraws from the surface of the half-space) a sudden interruption of the interaction occurs (the interaction force falls to zero) when $c_i = c^l$ (curve 2). We will denote the dependence of the force on the depth of the penetration, corresponding to curve 2, by $q_i^l(c_i)$. Hence, the processes in which the asperity approaches the surface of the half-space and moves away from it are represented by different curves, which corresponds to different values of the force q_i, the radius of the contact area

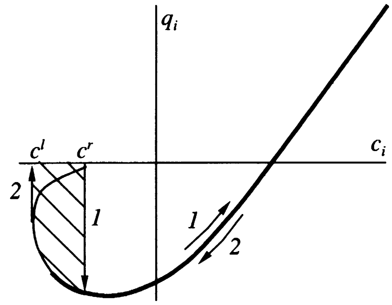

Fig. 3.

a_i and other characteristics for the same value of the penetration c_i. The functions $q_i^r(c_i)$ and $q_i^l(c_i)$ are found from relations (2.8)–(2.10), taking into account the choice of the necessary branch of the non-unique dependence of the load on the depth of penetration.

The solution for a rough cylinder. We will apply the solution obtained above for the interaction of a single asperity with an elastic half-space to the problem of the rolling of a rough cylinder. For a specified maximum depth of penetration of the cylinder into the half-space c, which is identical with the depth of penetration of the central asperity, the depths of penetration of the remaining asperities c_i are given by relation (2.1). The solution obtained above enables us, from these quantities, to determine the values of the normal forces q_i, acting on each asperity from the side of the elastic half-space (as can be seen from Fig. 3, these forces can be positive or negative depending on the values of c_i) and other characteristics of the contact interaction – the radii of the contact area a_i and the regions of adhesive interaction b_i.

The solution for the i-th asperity will depend on the depth of penetration c_i and on whether this asperity approaches the elastic half-space or moves away from it: if it approaches, the force acting on the asperity is determined by the function $q_i^r(c_i)$; if it is moving away from it, the force is determined by the function $q_i^l(c_i)$. In a similar way we can find the radius of the contact area and other characteristics.

The number of asperities interacting with the half-space on the right of the central asperity (the asperities approaching the elastic half-space), is determined from a relation similar to (2.1)

$$c - c^r = 2R\sin^2\frac{N^r l}{2R}$$

whence we have

$$N^r = \left[\frac{l}{2R}\arcsin\sqrt{\frac{c-c^r}{2R}}\right] \tag{2.11}$$

We can similarly determine the number of asperities which interact with the half-space on the left of the central asperity

$$N^l = \left[\frac{l}{2R}\arcsin\sqrt{\frac{c-c^l}{2R}}\right] \tag{2.12}$$

In expressions (2.11) and (2.12) $[x]$ denotes the greatest integer not exceeding x. Since $c^r \geq c^l$ we have $N^r \leq N^l$. In cases when the depth of penetration of the central asperity is so small that $c < c^l$ or $c < c^r$, the number of asperities which interact with the half-space from the corresponding side of the central asperity, is equal to zero, i.e. $N^l = 0$ or $N^r = 0$ respectively.

Hence, for a specified indentation of the central asperity c we can calculate the penetration of all the asperities from relations (2.1), and then, from Eqs. (2.8) and (2.9), we can determine how the force q_i acting on each asperity depends on the depth of penetration of this asperity c_i. From an analysis of this relation we can determine the quantities c^r and c^l, and also the functions $q_i^r(c_i)$ and $q_i^l(c_i)$. In a similar way we can find how the dimensions of the area of contact of the asperity with the half-space a_i^r and a_i^l depend on the depth of penetration of the asperity c_i. The functions obtained enable us to calculate the values of the forces acting on each asperity: on the right (q_i^r, $i = 1..N^r$) and on the left

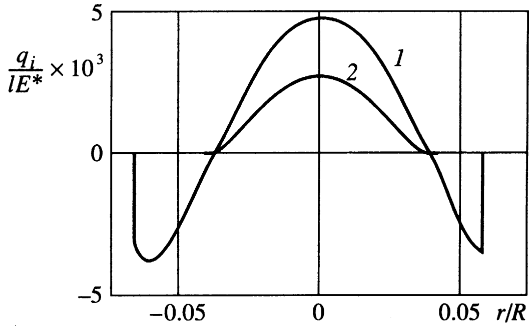

Fig. 4.

(q_i^l, $i = 1..N^l$) of the central asperity. We can calculate the force acting on the central asperity from any of the functions $q_0 = q_0^r(c) = q_0^l(c)$ obtained. The normal force acting on the cylinder is given by the relation

$$P = q_0 + \sum_{i=1}^{N^r} q_i^r + \sum_{i=1}^{N^l} q_i^l \tag{2.13}$$

3. Calculation of the contact characteristics

The results of a calculation of the forces acting on a asperities of the cylinder are shown in Fig. 4, where curve *1* represents the distribution of the average dimensionless pressure $q_i/(lE^*)$ as a function of the dimensionless coordinate $x_i/R = li/R$. The results were obtained for a dimensionless surface energy $\gamma/(p_0\rho) = 0.01$, a dimensionless adhesive pressure $p_0/E^* = 0.1$ and a dimensionless normal force acting on the cylinder $P/(E^*R^2) = 1.25 \times 10^{-4}$. The ratio of the radii of the asperities and the cylinder $\rho/R = 0.01$, and the number of asperities of the cylinder cross section $N = 5 \times 10^3$. The calculations showed that in this case the number of asperities, interacting with the half-space on the right, is equal to $N^r = 46$, and on the left $N^l = 53$. It can be seen that the average pressures are positive in the central part of the region of interaction of the cylinder with the half-space and negative at its edges. Moreover, the pressures are distributed asymmetrically, which leads to the occurrence of a moment that resists the rolling of the cylinder. For comparison we show a graph of the distribution of the average contact pressures when there is no adhesion (curve 2); in this case the pressure is everywhere positive, the distribution is symmetrical and there is no resistance to rolling.

The solution obtained also enables us to calculate the actual contact area S_r (the sum of the areas of all contact spots) and the nominal contact area S_n (the minimum area of the convex figure containing all the contact spots), per unit length l of the cylinder

$$S_r = \pi\left(a_0^2 + \sum_{i=1}^{N^r} (a_i^r)^2 + \sum_{i=1}^{N^l} (a_i^l)^2\right), \quad S_n = l^2(1 + N^r + N^l)$$

In Fig. 5 we show graphs of the dimensionless nominal contact area S_n/R^2 (the dashed curves) and the actual contact area S_r/R^2 (the continuous curves) as a function of the normal force acting on the cylinder $P/(E^*R^2)$ when there is adhesion with parameters $\gamma/(p_0\rho) = 0.01$ and $p_0/E^* = 0.1$ (curves 1) and when there is no adhesion (curves 2). The geometrical characteristics of the cylinder are $\rho/R = 0.01$ and $N = 10^4$. The results show that taking adhesion into account leads to a nonunique dependence on the load of both the nominal and the actual contact areas, and also to the existence of contact in a certain region of negative loads on the cylinder. The solution of the problem when the adhesive interaction of the surfaces is ignored gives reduced values of both the nominal and actual contact areas.

4. Calculation of the resistance to rolling

As follows from the solution of the problem considered in Section 2 for an individual asperity, the relation between the force acting on one asperity q_i and the depth of penetration of this asperity c_i (Fig. 3) is non-unique. It follows from

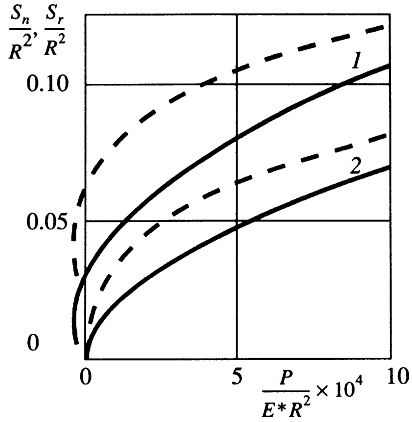

Fig. 5.

this that for a cyclical approach of the asperity to a half-space and its separation from it a loss of energy occurs provided that the greatest depth of penetration of the asperity per cycle exceeds c^r. The value of this energy loss corresponds to the area shown hatched in Fig. 3 and is given by the expression

$$\Delta w = \int_{c^r}^{c^l} [q_i^r(c) - q_i^l(c)]dc \tag{4.1}$$

The energy loss in a complete rotation of the cylinder is $\Delta w N_1$, where N_1 is the number of asperities in a section of the cylinder, for which the maximum depth of penetration into the half-space after a single rotation of the cylinder exceeds c^r. Assuming that this energy loss is equal to the work of the moment of the resistance to rolling per single rotation of the cylinder $2\pi M$, we obtain the following expression for the moment of the resistance

$$M = N_1 \Delta w/(2\pi) \tag{4.2}$$

In the model of a rough cylinder considered in the previous sections, which has N similar asperities in a section, the number N_1 is given by a step function

$$N_1 = \begin{cases} N, & c \geq c^r \\ 0, & c < c^r \end{cases} \tag{4.3}$$

We can also consider the case when the asperities have a statistical distribution over the height

$$N_1 = N \int_{-\infty}^{c} \varphi(t)dt \tag{4.4}$$

where $\varphi(t)$ is the distribution density; for example, for Gauss' law

$$\varphi(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

Graphs of the dimensionless moment of the resistance to rolling $M/(E^* R^3)$ as a function of the distance between the cylinder and the half-space $d = -c$, obtained from the calculations, are shown in Fig. 6 for a cylinder with the single-level model of roughness (curve 1) and a Gaussian distribution of the asperities over the height (curve 2). The results were obtained for

$$\frac{\gamma}{p_0 \rho} = 0.1, \quad \frac{p_0}{E^*} = 0.1, \quad \frac{\rho}{R} = 0.01, \quad N = 10^4$$

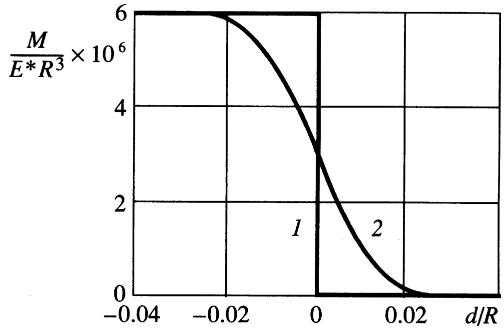

Fig. 6.

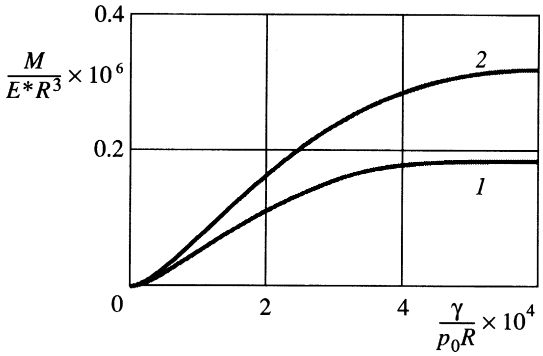

Fig. 7.

The root mean square deviation in the case of a Gaussian distribution of the heights $\sigma/R = 0.01$. The results show that, when the depth of penetration increases, the resistance to rolling increases in the first case abruptly, while in the second it occurs smoothly and approaches a constant value, which depends on the geometrical parameters of the cylinder, the elastic properties of the half-space and the characteristics of the adhesive interaction potential.

The results also show that the moment of the resistance to rolling $M/(E^* R^3)$ increases as the surface energy of the interacting bodies γ increases. This is illustrated by the curves shown in Fig. 7, which were obtained for $p_0/E^* = 0.1$, $N = 10^4$ and various radii of curvature of the asperity: $\rho/R = 0.01$ (curve 1) and $\rho/R = 0.012$ (curve 2) for the same roughnesses and depths of penetration over the height, exceeding c^r. It can be seen that the moment of the resistance when the surface energy increases reaches a constant value, which is a consequence of the use of the Maugis-Dugdale model for adhesive attraction of the surfaces. It also follows from the results that adhesion losses of energy and, consequently, the resistance to rolling are greater for the case of asperities of large radius ρ.

5. Conclusions

We have constructed a model of the rolling of a rough cylinder on an elastic half-space when there is adhesive attraction between the surfaces.

We have calculated the contact characteristics: the distribution of the average pressures on elementary contact spots, and the nominal and actual contact areas.

We have established that, for asperities which approach the half-space and which separate from it during rolling, there are different dependences of the force acting on the asperity from the side of the half-space, the size of the contact area and other contact characteristics on the depth of penetration of the asperity into the half-space. This leads to the occurrence of a moment of resistance to rolling of the cylinder.

We have proposed a method of determining the moment of adhesive resistance to rolling of the cylinder, based on a calculation of the energy loss when each asperity approaches the half-space and separates from it during the rolling of the cylinder.

We have analysed the moment of the resistance to rolling as a function of the surface energy of the interacting bodies and the depth of penetration of the cylinder into the elastic half-space, and also for different models of the roughness of the cylinder: single-level and with a Gaussian distribution of the asperities over the height for different radii of the asperities.

Acknowledgement

This research was supported financially by the Russian Foundation for Basic Research (05-08-18204, 07-01-00282 and 06-01-81021-Bel_a).

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Translated by R.C.G.